Lecture 11

- Review
- Big picture for evolution
- Cosmological perturbations in GR

Big picture:

- → We will study linearized Einstein equations for perturbations on the metric and various matter components around FRW background, we will work mostly in Fourier space.

- → It will be important to distinguish modes that are inside and outside of the cosmological horizon

- → Important are the perturbations that do not decay with time

- → Prior to recombination perturbations mostly oscillate ~ observed in CMB anisotropies.

- → After recombination growth is the most important feature ~ observed in large scale structure.

Cosmological Perturbations in GR

FRW reminder and conformal time

$$ds^2 = a^2(\eta)\left[-d\eta^2 + dx^i dx^i\right]$$

$$a(\eta)\, d\eta = dt$$

$$H = \frac{\dot{a}}{a} = \frac{a'}{a^2}$$

Friedmann eq:

$$\frac{a'^2}{a^4} = \frac{8\pi}{3} G \rho$$

$$2\frac{a''}{a^3} - \frac{a'^2}{a^4} = -8\pi G p$$

$$\rho' = -3\frac{a'}{a}(p+\rho)$$

RD: $a(\eta) = c \cdot \eta \simeq t^{1/2}$

MD: $a(\eta) = c \cdot \eta^2 \simeq t^{2/3}$

DE: $a = \frac{1}{H\eta} \simeq e^{Ht}$

$\Omega_{DM} = 0.22 \quad \Omega_B \simeq 0.05 \quad \Omega_\Lambda = 0.73$

$|\Omega_{curv}| < 0.01 \qquad z_{eq} \simeq 3000$

$$H = H_0 \cdot \sqrt{\Omega_\Lambda + \Omega_M (1+z)^3 + \Omega_{rad} (1+z)^4}$$

$$\eta = \int_z^\infty \frac{d\bar{z}}{a_0 H_0 \sqrt{\Omega_\Lambda + (1+\bar{z})^3 \Omega_M + (1+\bar{z})^4 \Omega_{rad}}}$$

$\frac{\eta_r}{\eta_{eq}} \simeq 2.4 \qquad \frac{\eta_0}{\eta_{eq}} \simeq 1.2 \cdot 10^2$

$\frac{\eta_0}{\eta_r} \sim 50$

Gauge and perturbations

$$ds^2 = a^2(\eta)\,\gamma_{\mu\nu}\,dx^\mu dx^\nu$$

$$\gamma_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

$$T^\mu{}_\nu = \bar{T}^\mu{}_\nu(\eta) + \delta T^\mu{}_\nu(x)$$

Einstein eq for perturbations:

$$\delta G^\mu{}_\nu = 8\pi G\, \delta T^\mu{}_\nu$$

$$\delta\left(\nabla_\mu T^\mu{}_\nu\right) = 0$$

We raise indices of perturbations with $\eta_{\mu\nu}$:

e.g. $$h^\mu{}_\nu = \eta^{\mu\lambda} h_{\lambda\nu}$$

$$\gamma^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

$$g^{\mu\nu} = \frac{1}{a^2}\left(\eta^{\mu\nu} - h^{\mu\nu}\right)$$

Diffeomorphisms:

$$g^{\mu\nu} \longrightarrow \tilde{g}^{\mu\nu} + \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu$$

$$\tilde{h}^{\mu\nu} = h^{\mu\nu} - \partial^\mu \xi^\nu - \partial^\nu \xi^\mu -$$

$$- 2\eta^{\mu\nu} \xi^\lambda \frac{\partial_\lambda a}{a} \qquad [h \sim \xi \sim \text{small}]$$

Let's fix the gauge by imposing 4 gauge conditions on $h_{\mu\nu}$

$h_{0i} = 0$ is 3 conditions.

There is a residual gauge invariance, that we will use later.

- Energy-momentum tensor of ideal fluid:

$$T^\mu{}_\nu = (\hat{\rho} + \hat{p}) u^\mu u_\nu - \delta^\mu{}_\nu \hat{p}$$

$$\hat{p} = p + \delta p \qquad \hat{\rho} = \rho + \delta\rho$$

- 4-velocity u^μ

$$g_{\mu\nu} u^\mu u^\nu = -1 \qquad (*)$$

$$\bar{u}^0 = a^{-1}, \quad \bar{u}^i = 0$$

$$u^0 = \frac{1}{a}(1+\delta u^0) \qquad u^i = \frac{1}{a} v^i$$

substitute into $(*)$:

$$(1+h_{00})(1+\delta u^0)^2 = 1 + O(v^i v^i)$$

$$\delta u^0 = -\frac{1}{2} h_{00}$$

$$u_0 = -a^2(1+h_{00})u^0 = a\left(1+\frac{1}{2}h_{00}\right)$$

$$\delta T^0{}_0 = \delta\rho$$

$$\delta T^0{}_i = -(\rho+p)v_i$$

$$\delta T^i{}_j = -\delta^i{}_j\,\delta p \qquad (v_i = \delta_{ij} v^j)$$

- Derivation of 2.40
2.41

$$\delta\rho' + 3\frac{a'}{a}(\delta\rho + \delta p) + (\rho + p)\left(\partial_i v_i - \frac{1}{2}h'\right) = 0$$

$(h = h_{ii})$

$$\partial_i \delta p + (\rho + p)\left(4\frac{a'}{a}v_i + \frac{1}{2}\partial_i h_{00}\right) +$$

$$+\left[v_i(\rho + p)\right]' = 0$$

stress tensor conservation

Fourier transform and helicities

Background is translational invariant — ~ use Fourier transform for perturbations

$$h_{\mu\nu}(\eta, x) = \int d^3k\, e^{ikx} h_{\mu\nu}(k),$$

same for $\delta\rho$, δp, v_i

$$\partial_i \leftrightarrow ik_i$$

$$q(\eta) = \frac{k}{a(\eta)} \quad \text{(physical momentum)}$$

Modes of different momenta do not mix!

Background is also rotationally invariant. picking $\vec{k}$ leaves $SO(2)$ unbroken.

Scalar components:

- scalars
- vectors $\sim k_i$ $\quad (v_i \sim k_i)$
- tensors $\sim \delta_{ij}$, $k_i \cdot k_j$

Vector components:

- vector $\perp \vec{k}$ $\qquad v_i k^i = 0$

Tensor components:

- traceless tensor transverse to k

$$h^{TT}_{ii} = 0 \qquad k_i h^{TT}_{ij} = 0$$

Rotations do not mix different helicities.

$$h_{00} = 2\varphi$$

$$h_{ij} = -2\psi\delta_{ij} - 2k_ik_jE + i(k_iW_j^T + k_jW_i^T) + h_{ij}^{TT}$$

$$v_i = V_i^T + ik_i v$$

Scalar Perturbations

At the linear level perturbations of different helicities do not mix (same as for different momenta)

Scalar pert. are the most important (also hardest)

Now let's use the residual gauge freedom:

$$\xi_i = -\partial_i\sigma \qquad \xi_0 = \partial_\eta\sigma$$

(leaves $h_{0i} = 0$)

$$\tilde{h}_{ij} = h_{ij} - 2\partial_i\partial_j\sigma - 2\frac{a'}{a}\delta_{ij}\sigma'$$

We can chose σ to kill E.
This leaves the following scalar degrees of freedom:

$$h_{00} = 2\varphi \quad , \quad h_{ij} = -2\psi\delta_{ij}$$

$$v_i = \partial_i v \quad , \quad \delta\rho , \quad \delta p$$

$$ds^2 = a^2(\eta)\left(-(1+2\varphi)\,d\eta^2 + (1+2\psi)\,dx^2\right)$$

The Einstein equations reduce to

$$\psi = -\varphi$$

$$\Delta\varphi - 3\frac{a'}{a}\varphi' - 3\frac{a'^2}{a^2}\varphi = 4\pi G a^2 \delta\rho_{tot}$$

$$\varphi' + \frac{a'}{a}\varphi = -4\pi G a^2 \left[(\rho + p) v\right]_{tot}$$

$$\varphi'' + 3\frac{a'}{a}\varphi' + 2\left(\frac{a''}{a} - \frac{a'^2}{a^2}\right)\varphi = 4\pi G a^2 \delta p_{tot}$$

tot $\rightarrow$ because we include the possibility of having several components

We also slightly simplify the conservation equations:

$$\delta\rho_\lambda' + 3\frac{a'}{a}(\delta\rho_\lambda + \delta p_\lambda) + (\rho_\lambda + p_\lambda)\cdot$$
$$\cdot(\Delta v_\lambda - 3\Phi') = 0$$

$$\left[(\rho_\lambda + p_\lambda)v_\lambda\right]' + 4\frac{a'}{a}(\rho_\lambda + p_\lambda)v_\lambda +$$
$$+ \delta p_\lambda + (\rho_\lambda + p_\lambda)\Phi = 0$$

For each component λ

$$\delta p_\lambda = u_{s,\lambda}^2 \delta\rho_\lambda$$

Subhorizon and Superhorizon regimes

$$q(\eta) = \frac{k}{a(\eta)} \underset{\text{(super)}}{\ll \mathcal{H}(\eta)} \quad \text{or} \quad \underset{\text{(sub)}}{\gg \mathcal{H}(\eta)}$$

decelerated expansion:

$$a < c \cdot t \quad , \mathcal{H} \sim \frac{1}{t} \quad \Rightarrow$$

$$\Rightarrow \quad \frac{q}{H} \nearrow \quad \text{modes enter horizon}$$

accelerated modes exit.